Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 3, 1995

BENDING OF A THREE-LAYER ORTHOTROPIC BEAM

A. E. Alekseev

UDC 539.3

Following the classification of [1, 2], the principal approaches to the development of the theory of multilayer structures are divided into several trends. These are primarily studies based on kinematic hypotheses accepted either for a packet as a whole (the hypothesis of a unique normal) or for each individual layer (for example, the hypothesis of a broken line [3]).

The approach to the development of the theory of layered media reported in [4] in which the basic equations are derived using the energetic procedure of spreading or averaging is widely used.

The methods based upon presenting the characteristics of a stress-strained state as a series (asymptotic methods and the methods in which unknowns are expanded into the series with respect to the transvere coordinate) are used to elaborate more precise theories.

The present paper belongs to this trend. As a basis we took the results of [5, 6] where the method of deriving the equations of an elastic layer is formulated for arbitrary boundary conditions at face surfaces (either displacements or stresses can be assigned) while retaining the differential order of the appropriate system of differential equations. A distinctive feature of this approach is the use of several approximations for the same sought quantities.

The method is used in solving the problem on bending of a three-layer orthotropic beam. The elastic deformation of each monolayer is described by the equations in [7]. At the interlayer boundaries the conjugation conditions are satisfied both for displacements and stresses. The proficiency of the method is illustrated by the example of bending of a three-layer carbon-filled plastic beam. A comparison with the solutions available in the literature is made.

1. Equations for Elastic Deformation of an Orthotropic Beam. Let us consider a beam of unit width, height $2h$, and length L. The beam is deformed under the conditions of a plane stressed state σ_x, σ_y, σ_{xy}. We denote the displacements in the direction of the x and y axes by u and v, respectively. The external forces $\{p^+, q^+\}$ and $\{p^-, q^-\}$ are applied at the boundaries $y = h$ and $y = -h$ respectively ($p^\pm$ and $q^\pm$ are the normal and tangential forces). The beam is made of an orthotropic material whose orthotropy axes coincide with the x and y axes. The strains ε_x, ε_y, and ε_{xy} are related to the stresses by Hooke's law

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_{yx}\frac{\sigma_y}{E_y}, \quad \varepsilon_y = \frac{\sigma_y}{E_y} - \nu_{xy}\frac{\sigma_x}{E_x},$$
$$2\varepsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}}, \quad \frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}. \tag{1.1}$$

Following the results of [6], the stresses are approximated by a truncated series with respect to Legendre's polynomials $P_k(\xi)$ ($\xi = y/h$):

$$2h\sigma_x = N + \frac{3M}{h}P_1(\xi), \quad \sigma_y = p_0 + \Delta p P_1(\xi),$$
$$2h\sigma_{xy} = Q + 2h\Delta q\, P_1(\xi) + (2hq_0 - Q)P_2(\xi),$$
$$\Delta q = 0.5(q^+ - q^-), \quad q_0 = 0.5(q^+ + q^-),$$
$$\Delta p = 0.5(p^+ - p^-), \quad p_0 = 0.5(p^+ + p^-). \tag{1.2}$$

Institute of Hydrodynamics, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 36, No. 3, pp. 158–166, May–June, 1995. Original article submitted January 10, 1994, revision submitted July 11, 1994.

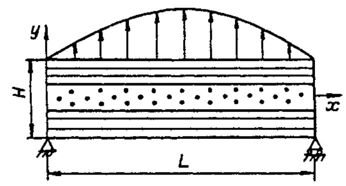

Fig. 1. Bending of a three-layer beam

Here $N = \int_{-h}^{h} \sigma_x \, dy$ is the force; $M = \int_{-h}^{h} \sigma_x y \, dy$ is the moment; $Q = \int_{-h}^{h} \sigma_{xy} \, dy$ is the shearing force.

The displacements and strains are approximated by a truncated series:

$$u = \sum_{k=0}^{3} [u]^k P_k(\xi), \quad v = \sum_{k=0}^{2} [v]^k P_k(\xi); \tag{1.3}$$

$$\begin{gathered} \varepsilon_x = \frac{d}{dx}[u]^0 + \frac{d}{dx}[u]^1 P_1(\xi), \quad \varepsilon_y = \frac{1}{h}[v]^1 + \frac{3}{h}[v]^2 P_1(\xi), \\ 2\varepsilon_{xy} = \frac{d}{dx}[v]^0 + \frac{1}{h}([u]^1 + [u]^3) + \frac{3}{h}[u]^2 P_1(\xi) + \frac{5}{h}[u]^3 P_2(\xi). \end{gathered} \tag{1.4}$$

Substituting the stresses (1.2) and stresses (1.4) in Hooke's law (1.1) and equating the coefficients of identical polynomials, we obtain

$$\begin{gathered} \frac{d}{dx}[u]^0 = \frac{N}{2hE_x} - \nu_{xy}\frac{p_0}{E_y}, \quad \frac{d}{dx}[u]^1 = \frac{3M}{2h^2 E_x} - \nu_{xy}\frac{\Delta p}{E_y}, \\ \frac{d}{dx}[v]^0 + \frac{1}{h}([u]^1 + [u]^3) = \frac{Q}{2hG_{xy}}; \end{gathered} \tag{1.5}$$

$$\begin{gathered} [u]^2 = \frac{h}{3G_{xy}}\Delta q, \quad [u]^3 = \frac{h}{5G_{xy}}\Big(q_0 - \frac{Q}{2h}\Big), \\ [v]^1 = h\frac{p_0}{E_y} - \nu_{xy}\frac{N}{2E_x}, \quad [v]^2 = h\frac{\Delta p}{3E_y} - \nu_{xy}\frac{M}{2hE_x}. \end{gathered} \tag{1.6}$$

We add the equilibrium equations to Eqs. (1.5) and (1.6):

$$\frac{d}{dx}N + 2\Delta q = 0, \quad \frac{d}{dx}M - Q + 2hq_0 = 0, \quad \frac{d}{dx}Q + 2\Delta p = 0. \tag{1.7}$$

Given the external forces $\{p^+, q^+\}$, $\{p^-, q^-\}$, Eqs. (1.5) and (1.7) form a closed system of ordinary differential equations of the sixth order with respect to the unknown functions N, M, Q, $[u]^0$, $[u]^1$, and $[v]^0$. The unknown functions $[u]^2$, $[u]^3$, $[v]^1$, and $[v]^2$ are determined from algebraic equations (1.6).

2. Equations of Bending of a Three-Layer Beam. Let us consider the problem on bending of a three-layer beam of thickness H and length L (Fig. 1), composed of reinforced monolayers. The directions of reinforcement of the first and third layers coincide with the beam axis, while the reinforcement of the second layer is directed perpendicular to the beam axis: $E_x^k = E_{11}$, $G_{xy}^k = G_{12}$ $(k = 1, 3)$; $E_x^2 = E_{22}$, $G_{xy}^2 = G_{23}$. Hereafter all values relevant to the kth layer are denoted by the superscript k. The thickness of each monolayer is constant and equals $2h$ $(H = 6h)$.

At the face surfaces the beam is loaded by the external forces

$$p^{3+} = g_0 \sin\Big(\frac{\pi x}{L}\Big), \quad q^{3+} = 0, \quad p^{1-} = 0, \quad q^{1-} = 0. \tag{2.1}$$

The elastic deformation of each monolayer is described by Eqs. (1.5)–(1.7). Three local reference frames are introduced (x^k, y^k):

$$x^k = x \quad (k=1,2,3), \quad y^1 = H/3 + y, \quad y^2 = y, \quad y^3 = y - H/3.$$

Accordingly, the dimensionless variable is $\xi = y^k/h \quad (k=1,2,3)$.

To simplify the problem let us assume that the displacements v^k $(k=1,2,3)$ are independent of the coordinate y, i.e., in the approximations of displacements (1.3) and strains (1.4) we have

$$[v^k]^\alpha = 0, \quad (k=1,2,3, \quad \alpha = 1,2). \tag{2.2a}$$

Furthermore, in Hooke's law we assume that

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y} \cong 0. \tag{2.2b}$$

Under these assumptions Eqs. (1.5)–(1.7) for each monolayer take the form

$$\begin{gathered}\frac{d}{dx}N^k + 2\Delta q^k = 0, \quad \frac{d}{dx}M^k - Q^k + 2hq_0^k = 0, \quad \frac{d}{dx}Q^k + 2\Delta p^k = 0.\\ \frac{d}{dx}[u^k]^0 = \frac{N^k}{2hE_x^k}, \quad \frac{d}{dx}[u^k]^1 = \frac{3M^k}{2h^2E_x^k},\\ \frac{d}{dx}[v^k]^0 + \frac{1}{h}([u^k]^1 + [u^k]^3) = \frac{Q^k}{2hG_{xy}^k},\\ [u^k]^2 = \frac{h}{3G_{xy}^k}\Delta q^k, \quad [u^k]^3 = \frac{h}{5G_{xy}^k}\left(q_0^k - \frac{Q^k}{2h}\right) \quad (k=1,2,3).\end{gathered} \tag{2.3}$$

At the surfaces of the interlayer contact the conditions of continuity of stresses

$$q^{1+} = q^{2-}, \quad q^{2+} = q^{3-}, \quad p^{1+} = p^{2-}, \quad p^{2+} = p^{3-} \tag{2.4}$$

and displacements

$$\begin{gathered}\sum_{n=0}^{3}[u^1]^n = \sum_{n=0}^{3}(-1)^n[u^2]^n, \quad \sum_{n=0}^{3}[u^2]^n = \sum_{n=0}^{3}(-1)^n[u^3]^n,\\ [v^1]^0 = [v^2]^0, \quad [v^2]^0 = [v^3]^0,\end{gathered} \tag{2.5}$$

should hold. Formulas (2.5) are obtained in terms of equalities (1.3) and (2.2a) and the properties of Legendre's polynomials.

The boundary conditions at the beam edges are given by the equalities

$$N^k = 0, \quad M^k = 0, \quad [v^k]^0 = 0 \quad (k=1,2,3), \quad \text{for} \quad x = 0, L. \tag{2.6}$$

As a result, the problem on elastic deformation of a three-layer beam is reduced to solving the system of ordinary differential equations (2.3)–(2.5) subject to boundary conditions (2.6), whose solution is of the form

$$\begin{gathered}q^{1+} = -\beta_1\lambda^3C(x), \quad q^{2+} = -\beta_2\lambda^3C(x),\\ p^{1+} = \alpha_1\lambda^4S(x), \quad p^{2+} = \alpha_2\lambda^4S(x), \quad \lambda = \pi/L.\end{gathered} \tag{2.7}$$

Here $\alpha_\gamma, \beta_\gamma$ are unknown constants; $C(x) = g_0\cos(\lambda x)$; $S(x) = g_0\sin(\lambda x)$. We integrate the system of equations (2.3) taking into account (2.4), (2.7) and boundary conditions (2.6). This yields the expressions for stresses

$$N^k = n_k\lambda^2S(x), \; Q^k = q_k\lambda^3C(x), \; M^k = m_k\lambda^2S(x) \; (k=1,2,3), \tag{2.8}$$

where

$$n_1 = \beta_1; \; n_2 = \beta_2 - \beta_1; \quad n_3 = -\beta_2; \; q_1 = \alpha_1; \; q_2 = \alpha_2 - \alpha_1; \; q_3 = 1/\lambda^4 - \alpha_2;$$

$$m_1 = \alpha_1 + h\beta_1; \quad m_2 = \alpha_2 - \alpha_1 + h(\beta_1 + \beta_2); \quad m_3 = 1/\lambda^4 - \alpha_2 + h\beta_2.$$

and for displacements

$$[u^k]^0 = -A(x)Khn_k, \ [u^k]^1 = -3A(x)Km_k, \ [u^k]^2 = -A(x)K_{12}htn_k/3,$$
$$[u^k]^3 = -A(x)K_{12}tm_k/5, \ [v^k]^0 = B(x)(3Km_k + K_{12}t(q_k + m_k/5)) \ (k = 1, 3),$$
$$[u^2]^0 = -A(x)hn_2, \quad [u^2]^1 = -3A(x)m_2, \quad [u^2]^2 = -A(x)K_{23}htn_2/3, \tag{2.9}$$
$$[u^2]^3 = A(x)K_{23}tm_2/5, \quad [v^2]^0 = B(x)(3m_2 + K_{23}t(q_2 + m_2/5)).$$

Here

$$A(x) = \lambda C(x)/(2h^2E_2); \quad B(x) = S(x)/(2h^3E_2); \quad t = \lambda^2h^2; \tag{2.10}$$
$$K = E_2/E_1; \quad K_{12} = E_2/G_{12}, \quad K_{23} = E_2/G_{23}.$$

Substituting (2.9) into the conditions of conjugation of layers (2.5), we obtain the system of four algebraic equations with respect to the parameters α_1, α_2, β_1, and β_2. Without presenting the calculations, we write down the solution

$$\beta_1 = \beta_2, \quad \alpha_1 + \alpha_2 = 1/\lambda^4,$$
$$\beta_1 + \beta_2 = -(36K + 8.4t(K_{12} + KK_{23}) + 0.96t^2K_{12}K_{23})/(\Delta h\lambda^4),$$
$$\alpha_2 - \alpha_1 = (36K + 3K^2 + t(8.4K_{12} + 2.4KK_{12} + 5.2KK_{23}) + t^2K_{12}(0.56K_{23} + 0.6K_{12}))/(\Delta\lambda^4), \tag{2.11}$$
$$\Delta = 78K + 3K^2 + t(12.8K_{12} + 12K_{23} + 5.2KK_{12} + 13.2kk_{23}) + t^2(0.6K_{12}^2 + 1.92K_{12}K_{23} + 0.84K_{23}^2).$$

We determine from (2.9) the dimensionless bending W at the point of its maximum ($x = L/2$):

$$W = \frac{100E_2H^3}{g_0L^4}[v^2]^0 = \frac{108 \cdot 10^2}{L^4}((\alpha_2 - \alpha_1)(3 + 1.2tK_{23}) + h(\beta_1 + \beta_2)(3 + 0.2tK_{23})).$$

Hence, using (2.11), we obtain finally

$$W = \frac{108 \cdot 10^2}{\pi^4} \frac{9K^2 + tKC_1 + t^2C_2 + t^3C_3}{78K + 3K^2 + tD_1 + t^2D_2}, \tag{2.12}$$

where

$$C_1 = 18K_{23} + 15.6K_{12} + 3.6KK_{23};$$
$$C_2 = 1.8K_{12}^2 + 7.2K_{12}K_{23} + 1.2KK_{23}^2 + 6,24KK_{12}K_{23};$$
$$C_3 = 0.48K_{12}K_{23}(K_{23} + K_{12});$$
$$D_1 = 12K_{23} + 12.8K_{12} + 13.2KK_{23} + 5.2KK_{12};$$
$$D_2 = 0.6K_{12}^2 + 1.92K_{12}K_{23} + 0.84K_{23}^2.$$

Stresses, strains, and displacements are calculated from formulas (1.2)–(1.4). For example, for the dimensionless axial stresses we have

$$\bar{\sigma}_x^k \equiv \frac{\sigma_x^k}{g_0} = \frac{N^k}{2hg_0} + \frac{3M^k}{2h^2g_0}P_1(\xi) \quad (k = 1, 2, 3), \tag{2.13}$$

while for the dimensionless tangential stresses we obtain

$$\bar{\sigma}_{xy}^k \equiv \frac{\sigma_{xy}^k}{g_0} = \frac{Q^k}{2hg_0} + \frac{\Delta q^k}{g_0}P_1(\xi) + \Big(\frac{q_0^k}{g_0} - \frac{Q^k}{2hg_0}\Big)P_2(\xi) \quad (k = 1, 2, 3). \tag{2.14}$$

3. Equations Based on the Broken Line Hypothesis. Following this hypothesis, the approximations of stresses, strains, displacements are obtainable from approximations (1.2)–(1.4), assuming that

$$[u]^2 = [u]^3 = [v]^1 = [v]^2 = 0, \tag{3.1}$$

in the form

$$\begin{gathered}
2h\sigma_x = N + \frac{3M}{h}P_1(\xi), \quad \sigma_y = p_0 + \Delta p P_1(\xi), \\
2h\sigma_{xy} = Q + 2h\Delta q P_1(\xi) + (2hq_0 - Q)P_2(\xi), \\
\Delta q = 0.5(q^+ - q^-), \quad q_0 = 0.5(q^+ + q^-), \\
\Delta p = 0.5(p^+ - p^-), \quad p_0 = 0.5(p^+ + p^-), \\
u = \sum_{k=0}^{1}[u]^k P_k(\xi), \quad v = [v]^0, \quad \varepsilon_x = \frac{d}{dx}[u]^0 + \frac{d}{dx}[u]^1 P_1(\xi), \quad \varepsilon_y = 0, \\
2\varepsilon_{xy} = \frac{d}{dx}[v]^0 + \frac{1}{h}[u]^1.
\end{gathered} \tag{3.2}$$

The appropriate differential equations for each layer follow from Eqs. (2.3):

$$\begin{gathered}
\frac{d}{dx}N^k + 2\Delta q^k = 0, \quad \frac{d}{dx}M^k - Q^k + 2hq_0^k = 0, \quad \frac{d}{dx}Q^k + 2\Delta p^k = 0, \\
\frac{d}{dx}[u^k]^0 = \frac{N^k}{2hE_x^k}, \quad \frac{d}{dx}[u^k]^1 = \frac{3M^k}{2h^2E_x^k}, \quad \frac{d}{dx}[v^k]^0 + \frac{1}{h}[u^k]^1 = \frac{Q^k}{2hG_{xy}^k}.
\end{gathered} \tag{3.3}$$

At the surfaces of layer contact the continuity conditions for stresses

$$q^{1+} = q^{2-}, \quad q^{2+} = q^{3-}, \quad p^{1+} = p^{2-}, \quad p^{2+} = p^{3-} \tag{3.4}$$

and displacements

$$\begin{gathered}
\sum_{n=0}^{1}[u^1]^n = \sum_{n=0}^{1}(-1)^n[u^2]^n, \quad \sum_{n=0}^{1}[u^2]^n = \sum_{n=0}^{1}(-1)^n[u^3]^n, \\
[v^1]^0 = [v^2]^0, \quad [v^2]^0 = [v^3]^0
\end{gathered} \tag{3.5}$$

should hold. The boundary conditions at the beam ends are identical to conditions (2.6):

$$N^k = 0, \quad M^k = 0, \quad [v^k]^0 = 0 \quad (k = 1,2,3) \quad \text{при} \quad x = 0. \tag{3.6}$$

Thus, with the use of the broken line hypothesis the problem of elastic deformation of a three-layer beam is reduced to the solution of a system of ordinary differential equations (3.3)–(3.5) subject to boundary conditions (3.6). After integration, we obtain the expressions for forces and moments

$$N^k = n_k\lambda^2 S(x), \; Q^k = q_k\lambda^3 C(x), \; M^k = m_k\lambda^2 S(x) \quad (k = 1,2,3); \tag{3.7}$$

where

$$\begin{gathered}
n_1 = \beta_1; \; n_2 = \beta_2 - \beta_1; \; n_3 = -\beta_2; \; q_1 = \alpha_1; \; q_2 = \alpha_2 - \alpha_1; \; q_3 = 1/\lambda^4 - \alpha_2; \\
m_1 = \alpha_1 + h\beta_1; \quad m_2 = \alpha_2 - \alpha_1 + h(\beta_1 + \beta_2); \quad m_3 = 1/\lambda^4 - \alpha_2 + h\beta_2,
\end{gathered}$$

and for displacements

$$\begin{gathered}
[u^k]^0 = -A(x)Khn_k, \quad [u^k]^1 = -3A(x)Km_k, \\
[v^k]^0 = B(x)(3Km_k + K_{12}tq_k) \quad (k = 1,3), \\
[u^2]^0 = -A(x)hn, \quad [u^2]^1 = -3A(x)m_2, \quad [v^2]^0 = B(x)(3m_2 + K_{23}tq_2).
\end{gathered} \tag{3.8}$$

Here

$$A(x) = \lambda C(x)/(2h^2E_2); \quad B(x) = S(x)/(2h^3E_2); \quad t = \lambda^2h^2;$$

$$K = E_2/E_1; \quad K_{12} = E_2/G_{12}; \quad K_{23} = E_2/G_{23}.$$

Substituting (3.8) into the conditions of conjugation of layers (3.6), we obtain a system of four algebraic equations with respect to the parameters α_1, α_2, β_1, β_2. Omitting the calculations, let us write the solution of the system as

$$\begin{gathered}\beta_1 = \beta_2, \quad \alpha_1 + \alpha_2 = 1/\lambda^4, \\ \beta_1 + \beta_2 = -(36K + 6t(K_{12} + KK_{23}))/(\Delta h \lambda^4), \\ \alpha_2 - \alpha_1 = (36K + 3K^2 + t(6K_{12} + 4KK_{12}))/(\Delta\lambda^4), \\ \Delta = 78K + 3K^2 + t(6K_{12} + 12K_{23} + 8KK_{12} + 4KK_{23}).\end{gathered} \tag{3.9}$$

Similar to (2.12), one can calculate the value of the dimensionless bending W at the point of its maximum ($x = L/2$):

$$W = \frac{100E_2H^3}{g_0L^4}[v^2]^0 = \frac{108 \cdot 10^2}{L^4}((\alpha_2 - \alpha_1)(3 + tK_{23}) + 3h(\beta_1 + \beta_2)).$$

Hence, using (3.9), we obtain finally

$$W = \frac{108 \cdot 10^2}{\pi^4} \frac{9K^2 + tK\tilde{C}_1 + t^2\tilde{C}_2}{78K + 3K^2 + t\tilde{D}_1} \tag{3.10}$$

where

$$\begin{gathered}\tilde{C}_1 = 18K_{23} + 12K_{12} + 3KK_{23}; \quad \tilde{C}_2 = 6K_{12}K_{23} + 4KK_{12}K_{23}; \\ \tilde{D}_1 = 12K_{23} + 6K_{12} + 8KK_{23} + 4KK_{12}.\end{gathered}$$

4. Comparison with Known Solutions. Let us consider as an example the solution of the problem on bending of a three-layer beam composed of carbon-plastic unidirectional reinforced monolayers with the parameters [8] $E_{11} = 1.724 \cdot 10^5$ mPa, $E_{22} = 6895$ mPa, $G_{12} = 3448$ mPa, $G_{23} = 1379$ mPa, ν_{12} =0.25. The geometry and the load on the beam are as follows: $H/L = 1/4$ и $1/10$, $g_0 = 0.6895$ mPa.

For the given values of elastic constants we get

$$K = E_2/E_1 \cong 4 \cdot 10^{-2}, \quad K_{12} = E_2/G_{12} \cong 2, \quad K_{23} = E_2/G_{23} \cong 5. \tag{4.1}$$

It should be noted that in the axial direction the elastic properties of the monolayers composing the beam differ substantially. It follows from (4.1) that to 10^{-2} one can take $1 \cong 1 + K$. Then we calculate the values of the parameter $t = \lambda^2h^2 = (\pi H/6L)^2$: with $H/L = 1/4$, $t = 1.71347 \cdot 10^{-2}$, while with $H/L = 1/10$, $t = 0.27416 \cdot 10^{-2}$. As for the parameter K, with the same order of accuracy we can take $1 \cong 1 + t$. As a result, formula (2.12) is simplified and reduced to the form

$$\begin{gathered}W = \frac{108 \cdot 10^2}{\pi^4} \frac{45K^2 + tKC_1^* + t^2C_2^*}{390K + tD^*}, \\ C_1^* = 90K_{23} + 78K_{12}, \quad C_2^* = 9K_{12}^2 + 36K_{12}K_{23}, \quad D^* = 15K_{23} + 16K_{12}.\end{gathered} \tag{4.2}$$

In a similar way we find from (3.10) for the case of the broken line hypothesis

$$\begin{gathered}W = \frac{108 \cdot 10^2}{\pi^4} \frac{45K^2 + tK\tilde{C}_1^* + t^2\tilde{C}_2^*}{390K + t\tilde{D}^*}, \\ \tilde{C}_1^* = 90K_{23} + 60K_{12}, \quad \tilde{C}_2^* = 30K_{12}K_{23}, \quad \tilde{D}^* = 60K_{23} + 30K_{12}.\end{gathered} \tag{4.3}$$

Table 1 presents the values of W and ε obtained on the basis of different theories: solution using the theory of elasticity [9], analytical solutions [8] using the theory of layered beams under the hypothesis of straight normals (classical theory), solutions based on the Timoshenko hypothesis and on the broken line hypothesis [formula (4.3)], and solutions using the finite elements method [8, 10] for different types of elements.

TABLE 1

$\frac{L}{H}$	Deflection W / Error ε, %	Elasticity theory [9]	Classical theory [8]	Timoshenko theory [8]	Finite elements method [8]	Finite elements method [10]	Formula (4.2)	Formula (4.3)
4	W	2.8872	0.5096	2.0943	3.5000	2.9102	2.9182	2.8020
	ε	—	82.35	27.46	−21.22	−0.80	−1.07	8.52
10	W	0.9316	0.5096	0.7631	0.9886	0.9317	0.9348	0.9135
	ε	—	42.29	18.09	−6.11	−0.003	−0.34	1.95

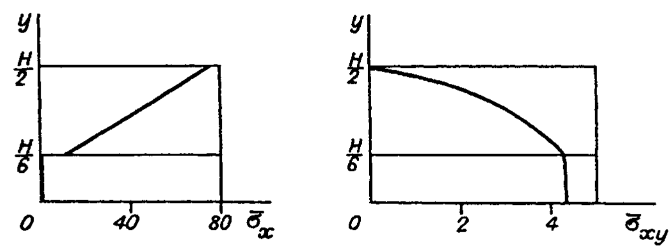

Fig. 2. Distribution of normal $\bar{\sigma}_x(L/2, y)$ and tangential stresses $\bar{\sigma}_{xy}(0, y)$ along the beam height with $L/H = 10$

The error ε is calculated as follows:

$$\varepsilon = \frac{W_{\mathrm{T}} - W_*}{W_{\mathrm{T}}} \cdot 100\%$$

(W_{T} is the exact value of the dimensionless bending obtained using the theory of elasticity, W_* is the approximate solution).

Analysis of the results presented in the table show that both for beams of average length ($L/H = 10$) and for short ones ($L/H = 4$) the bending calculated from formula (4.2) is in good agreement with the exact solution.

For very thin beams, i.e., when $t \to 0$, we have from (4.2) for the dimensionless bending W

$$W \to W_0 = \frac{972 \cdot 10^2}{78 \cdot \pi^4} K \cong 0.51172.$$

The quantity W_0 almost coincides with that obtained with the classical theory.

The distributions of normal $\bar{\sigma}_x^k$ and tangential $\bar{\sigma}_{xy}^k$ stresses calculated from formulas (2.13) and (2.14) also agree well with the solutions obtained with the theory of elasticity. By way of example, let us consider sections of the beam with $x = L/2$ and 0. In the first case, the axial stresses σ_x achieve maximum values; in the second case, the tangential ones σ_{xy}. It follows from (2.13) and (2.14) that

$$\bar{\sigma}_x^1(L/2, y) = \lambda^2(h\beta_1 + 3(\alpha_1 + h\beta_1)P_1(\xi))/2h^2,$$
$$\bar{\sigma}_x^2(L/2, y) = \lambda^2(h(\beta_2 - \beta_1) + 3(\alpha_2 - \alpha_1 + h(\beta_2 + \beta_1))P_1(\xi))/2h^2,$$
$$\bar{\sigma}_x^3(L/2, y) = \lambda^2(-h\beta_1 + 3(\alpha_1 + h\beta_1)P_1(\xi))/2h^2,$$

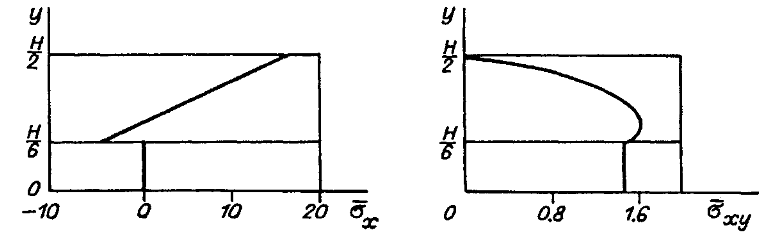

Fig. 3. Distribution of normal $\bar{\sigma}_x(L/2, y)$ and tangential stresses $\bar{\sigma}_{xy}(0, y)$ along the beam height with $L/H = 4$

$$\bar{\sigma}^1_{xy}(0,y) = \alpha_1\lambda^3(P_0(\xi) - P_2(\xi))/2h - \beta_1\lambda^3(P_1(\xi) + P_2(\xi))/2,$$
$$\bar{\sigma}^2_{xy}(0,y) = (\alpha_2 - \alpha_1)\lambda^3 P_0(\xi)/2h - (\alpha_2 - \alpha_1 + h(\beta_2 + \beta_1))\lambda^3 P_2(\xi)/2h,$$
$$\bar{\sigma}^3_{xy}(0,y) = \alpha_1\lambda^3(P_0(\xi) - P_2(\xi))/2h + \beta_1\lambda^3(P_1(\xi) - P_2(\xi))/2,$$

where the coefficients α_1, α_2, β_1, and β_2 are found from (2.11).

The appropriate curves of stress distribution for different length-to-thickness ratios, which are shown in Figs. 2 and 3, almost coincide with those obtained with the elasticity theory [9].

This work was supported by the International Science Foundation (Grant KИ0000) and the Russian Foundation for Fundamental Research (Grant 94-02-04022).

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